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Complexity Review

Decision Problems

- **Decision problem**: assignment of inputs to Yes (1) or No (0)
- Inputs are either **No instances** or **Yes instances** (i.e. satisfying instances)

Problem	Decision
s-t Shortest Path	Does a given G contain a path from s to t with weight at most d?
Negative Cycle	Does a given G contain a negative weight cycle?
Longest Path	Does a given G contain a **simple** path with weight at least d?
Subset Sum	Does a given set of integers A contain a subset with sum S?
Tetris	Can you survive a given sequence of pieces?
Chess	Can a player force a win from a given board?
Halting problem	Does a given computer program terminate for a given input?

- **Algorithm/Program**: constant length code (working on a word-RAM with $\Omega(\log n)$-bit words) to solve a problem, i.e., it produces correct output for every input and the length of the code is independent of the instance size
- Problem is **decidable** if there exists a program to solve the problem in finite time

Decidability

- Program is finite constant string of bits, problem is function $p : \mathbb{N} \to \{0, 1\}$, i.e., infinite string of bits
- (# of programs $|\mathbb{N}|$, countably infinite) $\ll$ (# of problems $|\mathbb{R}|$, uncountably infinite)
- (Proof by Cantor's diagonal argument, probably covered in 6.042)
- Proves that most decision problems not solvable by any program (undecidable)
- e.g. the Halting problem is undecidable (many awesome proofs in 6.045)
- Fortunately most problems we think of are algorithmic in structure and are decidable

Decidable Decision Problems

R	problems decidable in finite time	'R' comes from recursive languages
EXP	problems decidable in exponential time $2^{n^{O(1)}}$	most problems we think of are here
P	problems decidable in polynomial time $n^{O(1)}$	efficient algorithms, the focus of this class

- These sets are distinct, i.e. **P** $\subsetneq$ **EXP** $\subsetneq$ **R** (via time hierarchy theorems, see 6.045)

Nondeterministic Polynomial Time (NP)

- **P** is the set of decision problems for which there is an algorithm A such that for every instance I of size n, A on I runs in poly(n) time and solves I correctly
- **NP** is the set of decision problems for which there is an algorithm V, a "verifier", that takes as input an instance I of the problem, and a "certificate" bit string of length polynomial in the size of I, so that:
  - V always runs in time polynomial in the size of I,
  - if I is a YES-instance, then there is some certificate c so that V on input (I, c) returns YES, and
  - if I is a NO-instance, then no matter what c is given to V together with I, V will always output NO on (I, c).
- You can think of the certificate as a proof that I is a YES-instance. If I is actually a NO-instance then no proof should work.

Problem	Certificate	Verifier
s-t Shortest Path	A path P from s to t	Adds the weights on P and checks if $\leq d$
Negative Cycle	A cycle C	Adds the weights on C and checks if < 0
Longest Path	A path P	Checks if P is a **simple** path with weight at least d
Subset Sum	A set of items A'	Checks if $A' \in A$ has sum S
Tetris	Sequence of moves	Checks that the moves allow survival

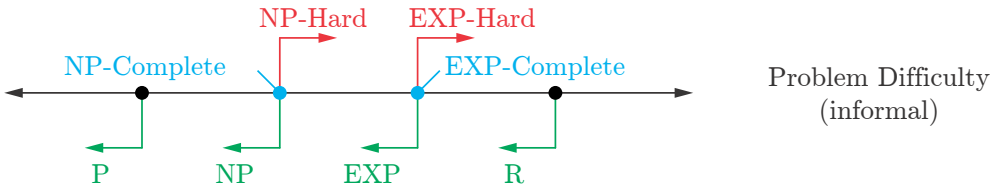
- **P** $\subset$ **NP**: the verifier V just solves the instance ignoring any certificate
- **NP** $\subset$ **EXP**: try all possible certificates! At most $2^{n^{O(1)}}$ of them, run verifier V on all of them
- **Open:** Does **P** $=$ **NP**? **NP** $=$ **EXP**?
- Most people think **P** $\subsetneq$ **NP** ($\subsetneq$ **EXP**), i.e., generating solutions harder than checking
- If you prove either way, people will give you lots of money. ($1M Millennium Prize)
- Why do we care? If can show a problem is hardest problem in **NP**, then problem cannot be solved in polynomial time if **P** $\neq$ **NP**
- How do we relate difficulty of problems? Reductions!

Reductions

- Suppose you want to solve problem A
- One way to solve is to convert A into a problem B you know how to solve
- Solve using an algorithm for B and use it to compute solution to A
- This is called a **reduction** from problem A to problem B ($A \to B$)
- Because B can be used to solve A, B is at least as hard ($A \leq B$)
- General algorithmic strategy: reduce to a problem you know how to solve

A	Conversion	B
Unweighted Shortest Path	Give equal weights	Weighted Shortest Path
Product Weighted Shortest Path	Logarithms	Sum Weighted Shortest Path
Sum Weighted Shortest Path	Logarithms	Product Weighted Shortest Path

- Problem A is **NP-Hard** if every problem in **NP** is polynomially reducible to A
- i.e. A is at least as hard as (can be used to solve) every problem in **NP** ($X \leq A$ for $X \in$ **NP**)
- **NP-Complete** = **NP** $\cap$ **NP-Hard**
- All **NP-Complete** problems are equivalent, i.e. reducible to each other
- First **NP-Complete** problem? Every decision problem reducible to satisfying a logical circuit, a problem called "Circuit SAT".
- Longest Path, Tetris are **NP-Complete**, Chess is **EXP-Complete**



Knapsack is NP-Hard

- Reduce known NP-Hard Problem to Knapsack: **Partition** known to be NP-Hard
  - Input: List of n numbers a_i
  - Output: Does there exist a partition into two sets with equal sum?
- Reduction: $s_i = v_i = a_i$, $s = \frac{1}{2}\sum a_i$

- Knapsack at least as hard as Partition, so since Partition is **NP-Hard**, so is Knapsack
- Knapsack in **NP**, so also **NP-Complete**